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IMPACT RESPONSE OF A CYLINDER COMPOSITE WITH A PENNY-SHAPED CRACK

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Abstract—The dynamic response of a penny-shaped crack in a cylinder composite under normal impact is analyzed. The crack is oriented normally to the interface. Laplace and Hankel transform techniques are used to reduce the elasto-dynamic problem to a pair of dual integral equations. The integral equations are solved by using an integral transform technique and the result is expressed in terms of a Fredholm integral equation of the second kind. A numerical Laplace inversion routine is used to recover the time dependence of the solution. The dynamic stress intensity factor is determined and its dependence on time, the material properties and the geometrical parameters are discussed.

INTRODUCTION

Fracture problems involving dissimilar materials weakened by crack-like imperfection have much attention because of the increasing interest in the analysis of composite structural failure under dynamic loading. Sih and Chen considered the dynamic response of a layered composite with a crack under normal and shear impact[1] or a penny-shaped crack under normal and radial impact[2]. The torsional impact response of a penny-shaped interface crack in a layered composite has been treated by Ueda *et al.*[3]. For a strip composite subjected to normal impact, the authors[4] have provided information on the dynamic behavior of a crack normal to the interfaces. This paper further applies the method of Sih and Chen to the case of a cracked cylinder embedded in a foreign material.

In this paper, the normal impact response of a cylinder composite containing a penny-shaped crack is investigated. The cylinder composite consists of a cylinder that is bonded to a different medium. The plane of the crack is perpendicular to the axis of the cylinder. Laplace and Hankel transforms are used to reduce the elasto-dynamic problem to a pair of dual integral equations. The dual integral equations are solved by an integral transform technique and the solution is expressed in terms of a Fredholm integral equation of the second kind having the kernel with finite integrals. A numerical Laplace inversion technique[5] is used to recover the time dependence of the solution. The dynamic stress intensity factor is computed and numerical values are shown in graphs for various material properties and geometrical parameters at designated time instances. As time becomes very large, all of the results here reduce to the corresponding static solutions[6].

FORMULATION OF THE PROBLEM

Let the cylinder composite shown in Fig. 1 be subjected to a time-dependent applied stress. The composite consists of an elastic cylinder of radius b with shear modulus μ_1, Poisson ratio ν_1, and mass density ρ_1 that is bonded to an infinite elastic medium of different material properties μ_2, ν_2, ρ_2. A cylindrical coordinate system (r, ϑ, z) is attached to the center of a penny-shaped crack of radius a that is symmetrically situated in the cylinder. Let the displacement components in the r, ϑ and z directions be denoted by u_r, u_ϑ and u_z,

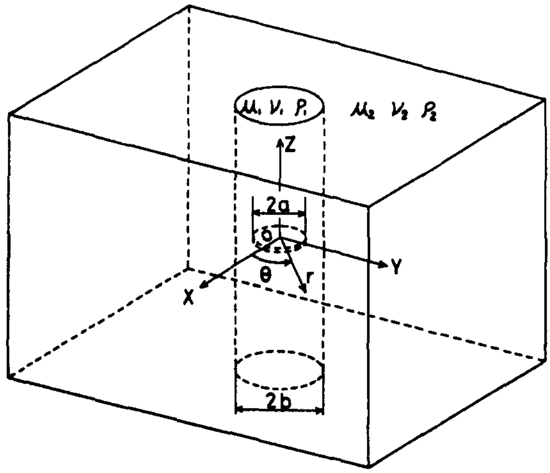

Fig. 1. A cylinder composite with a penny-shaped crack.

respectively. For an axially symmetric deformation field, the two nonzero displacement components can be expressed in terms of wave potentials $\varphi_j(r, \vartheta, z)$ and $\psi_j(r, \vartheta, z)$ as follows:

$$(u_r)_j = \frac{\partial \varphi_j}{\partial r} - \frac{\partial \psi_j}{\partial z},$$
$$(u_z)_j = \frac{\partial \varphi_j}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\psi_j), \tag{1}$$

where $j = 1$ refers to the cylinder with the crack and $j = 2$ to the surrounding material, and t is the time. The four nontrivial stress components are given by

$$(\sigma_r)_j = 2\mu_j \frac{\partial}{\partial r}\left(\frac{\partial \varphi_j}{\partial r} - \frac{\partial \psi_j}{\partial z}\right) + \lambda_j \nabla^2 \varphi_j,$$
$$(\sigma_\vartheta)_j = 2\mu_j \frac{1}{r}\left(\frac{\partial \varphi_j}{\partial r} - \frac{\partial \psi_j}{\partial z}\right) + \lambda_j \nabla^2 \varphi_j,$$
$$(\sigma_z)_j = 2\mu_j \frac{\partial}{\partial z}\left(\frac{\partial \varphi_j}{\partial z} + \frac{\partial \psi_j}{\partial r} + \frac{\psi_j}{r}\right) + \lambda_j \nabla^2 \varphi_j, \tag{2}$$
$$(\tau_{rz})_j = \mu_j\left[\frac{\partial}{\partial z}\left(2\frac{\partial \varphi_j}{\partial r} - \frac{\partial \psi_j}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{\partial \psi_j}{\partial r} + \frac{\psi_j}{r}\right)\right],$$

in which $\lambda_j = 2\mu_j\nu_j/(1-2\nu_j)$ and μ_j are the Lamé coefficients and ∇^2 represents the operator $\nabla^2 = \partial^2/\partial r^2 + (1/r)(\partial/\partial r) + \partial^2/\partial z^2$. The governing equations can thus be obtained from the equations of motion which yield

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi_j}{\partial r}\right) + \frac{\partial^2 \varphi_j}{\partial z^2} = \frac{1}{c_{1j}^2}\frac{\partial^2 \varphi_j}{\partial t^2},$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_j}{\partial r}\right) - \frac{\psi_j}{r^2} + \frac{\partial^2 \psi_j}{\partial z^2} = \frac{1}{c_{2j}^2}\frac{\partial^2 \psi_j}{\partial t^2}, \tag{3}$$

with $c_{1j} = [(\lambda_j + 2\mu_j)/\rho_j]^{1/2}$ and $c_{2j} = (\mu_j/\rho_j)^{1/2}$ being the dilatational and shear wave speeds.

Suppose that the penny-shaped crack is now loaded suddenly by a pair of normal stresses of magnitude $-\sigma_0$ such that the upper and lower crack surfaces move in the opposite directions. Therefore, the boundary conditions may be written as

$$(\tau_{rz})_1(r,0,t) = 0 \qquad (0 \leqslant r \leqslant b), \tag{4}$$

$$(\sigma_z)_1(r,0,t) = -\sigma_0 H(t) \qquad (0 \leqslant r < a),$$

$$(u_z)_1(r,0,t) = 0 \qquad (a \leqslant r \leqslant b), \tag{5}$$

where H( ) is the Heaviside unit step function. Perfect bonding will be assumed along the interface of material 1 and 2. The continuity conditions along $r = b$ are

$$(u_r)_1(b,z,t) = (u_r)_2(b,z,t), \tag{6}$$

$$(u_z)_1(b,z,t) = (u_z)_2(b,z,t), \tag{7}$$

$$(\sigma_r)_1(b,z,t) = (\sigma_r)_2(b,z,t), \tag{8}$$

$$(\tau_{rz})_1(b,z,t) = (\tau_{rz})_2(b,z,t). \tag{9}$$

METHOD OF SOLUTION

Define a Laplace transform pair by the equations

$$f^*(p) = \int_0^\infty f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t, \qquad f(t) = \frac{1}{2\pi i}\int_{Br} f^*(p)\,\mathrm{e}^{pt}\,\mathrm{d}p, \tag{10}$$

in which Br stands for the Bromwich path of integration. The application of the first equation of (10) to eqns (3) yields

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi_j^*}{\partial r}\right) + \frac{\partial^2 \varphi_j^*}{\partial z^2} &= \frac{p^2}{c_{1j}^2}\varphi_j^*, \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_j^*}{\partial r}\right) - \frac{\psi_j^*}{r^2} + \frac{\partial^2 \psi_j^*}{\partial z^2} &= \frac{p^2}{c_{2j}^2}\psi_j^*. \end{aligned} \tag{11}$$

The solutions to eqns (11) can be obtained as

$$\begin{aligned} \varphi_1^*(r,z,p) &= \int_0^\infty A_1(s,p)J_0(rs)\,\mathrm{e}^{-\gamma_{11}z}\,\mathrm{d}s \\ &\quad + \int_0^\infty A_2(s,p)I_0(\gamma_{11}r)\cos(sz)\,\mathrm{d}s, \\ \psi_1^*(r,z,p) &= \int_0^\infty B_1(s,p)J_1(rs)\,\mathrm{e}^{-\gamma_{21}z}\,\mathrm{d}s \\ &\quad + \int_0^\infty B_2(s,p)I_1(\gamma_{21}r)\sin(sz)\,\mathrm{d}s, \end{aligned} \tag{12}$$

$$\begin{aligned} \varphi_2^*(r,z,p) &= \int_0^\infty C_1(s,p)K_0(\gamma_{12}r)\cos(sz)\,\mathrm{d}s, \\ \psi_2^*(r,z,p) &= \int_0^\infty C_2(s,p)K_1(\gamma_{22}r)\sin(sz)\,\mathrm{d}s, \end{aligned} \tag{13}$$

where A_1, A_2, B_1, B_2 and C_1, C_2 are the unknowns to be determined, and the functions γ_{1j} and γ_{2j} are given by

$$\gamma_{1j} = \left(s^2 + \frac{p^2}{c_{1j}^2}\right)^{1/2}, \qquad \gamma_{2j} = \left(s^2 + \frac{p^2}{c_{2j}^2}\right)^{1/2}. \tag{14}$$

In eqns (12), (13), $J_n(\ )$, $I_n(\ )$ and $K_n(\ )$ are the Bessel functions of the first kind and the modified Bessel functions of the first and second kind of order n ($n = 0, 1$), respectively. In the Laplace transform domain, eqns (4)–(9) become

$$(\tau_{rz})_1^*(r,0,p) = 0 \qquad (0 \leqslant r \leqslant b), \tag{15}$$

$$(\sigma_z)_1^*(r,0,p) = -\sigma_0/p \qquad (0 \leqslant r < a),$$

$$(u_z)_1^*(r,0,p) = 0 \qquad (a \leqslant r \leqslant b), \tag{16}$$

$$(u_r)_1^*(b,z,p) = (u_r)_2^*(b,z,p), \tag{17}$$

$$(u_z)_1^*(b,z,p) = (u_z)_2^*(b,z,p), \tag{18}$$

$$(\sigma_r)_1^*(b,z,p) = (\sigma_r)_2^*(b,z,p), \tag{19}$$

$$(\tau_{rz})_1^*(b,z,p) = (\tau_{rz})_2^*(b,z,p). \tag{20}$$

Substituting eqns (12), (13) into (1), (2), one obtains the displacement and stress expressions in the Laplace transform plane. The satisfaction of eqns (15) and (16) by these expressions yields

$$\int_0^\infty A(s,p)J_0(rs)\,\mathrm{d}s = 0 \qquad (r \geqslant a), \tag{21}$$

$$\int_0^\infty sF(s,p)A(s,p)J_0(rs)\,\mathrm{d}s = -\frac{\sigma_0 c_{21}^2}{\mu_1(1-\kappa^2)p^3} - \frac{c_{21}^2}{p^2(1-\kappa^2)}\int_0^\infty \left\{\left(\frac{p^2}{c_{21}^2} - 2\gamma_{11}^2\right)I_0(\gamma_{11}r)A_2(s,p) + 2s\gamma_{21}I_0(\gamma_{21}r)B_2(s,p)\right\}\mathrm{d}s \qquad (r < a), \tag{22}$$

where $\kappa = c_{21}/c_{11}$,

$$A(s,p) = \frac{2\gamma_{11}}{s^2+\gamma_{21}^2}A_1(s,p) = \frac{1}{s}B_1(s,p), \tag{23}$$

$$F(s,p) = \frac{(s^2+\gamma_{21}^2)^2 - 4s^2\gamma_{11}\gamma_{21}}{2s\gamma_{11}(1-\kappa^2)}\left(\frac{c_{21}^2}{p^2}\right). \tag{24}$$

Through eqns (17)–(20), the unknowns $A_2(s,p)$ and $B_2(s,p)$ are related to the new parameter $A(s,p)$ by the equations

$$A_2(s,p) = \delta_1(s,p)\int_0^\infty g_1(\eta,p)A(\eta,p)\,\mathrm{d}\eta+\delta_2(s,p)\int_0^\infty g_2(\eta,p)A(\eta,p)\,\mathrm{d}\eta$$
$$+\delta_3(s,p)\int_0^\infty g_3(\eta,p)A(\eta,p)\,\mathrm{d}\eta+\delta_4(s,p)\int_0^\infty g_4(\eta,p)A(\eta,p)\,\mathrm{d}\eta,$$

$$B_2(s,p) = \delta_5(s,p)\int_0^\infty g_1(\eta,p)A(\eta,p)\,\mathrm{d}\eta+\delta_6(s,p)\int_0^\infty g_2(\eta,p)A(\eta,p)\,\mathrm{d}\eta$$
$$+\delta_7(s,p)\int_0^\infty g_3(\eta,p)A(\eta,p)\,\mathrm{d}\eta+\delta_8(s,p)\int_0^\infty g_4(\eta,p)A(\eta,p)\,\mathrm{d}\eta, \quad (25)$$

in which the quantities $g_i(\eta,p)$ $(i = 1\text{–}4)$, $\delta_i(s,p)$ $(i = 1\text{–}8)$ are given by eqns (A1), (A2) in Appendix A.

The set of dual integral equations (21) and (22) may be solved and the result is

$$A(s,p) = -\frac{2}{\pi}\frac{\sigma_0 c_{21}^2 a^2}{\mu_1(1-\kappa^2)p^3}\int_0^1 \Phi(\xi,p)\sin(sa\xi)\,\mathrm{d}\xi. \quad (26)$$

In eqn (26), the function $\Phi(\xi,p)$ is governed by the following Fredholm integral equation of the second kind:

$$\Phi(\xi,p)+\int_0^1 \{K_1(\xi,\eta,p)+K_2(\xi,\eta,p)\}\Phi(\eta,p)\,\mathrm{d}\eta = \xi. \quad (27)$$

The kernel functions $K_1(\xi,\eta,p)$ and $K_2(\xi,\eta,p)$ are given by

$$K_1(\xi,\eta,p) = \frac{2}{\pi}\int_0^\infty \left[F\left(\frac{s}{a},p\right)-1\right]\sin(s\xi)\sin(s\eta)\,\mathrm{d}s, \quad (28)$$

$$K_2(\xi,\eta,p) = \frac{2}{\pi}\int_0^\infty \left\{G_1\left(\frac{s}{a},p\right)\sinh(\gamma'_{11}\xi)\sinh(\gamma'_{11}\eta)\right.$$
$$+G_2\left(\frac{s}{a},p\right)\sinh(\gamma'_{21}\xi)\sinh(\gamma'_{11}\eta)+G_3\left(\frac{s}{a},p\right)\sinh(\gamma'_{11}\xi)\sinh(\gamma'_{21}\eta)$$
$$\left.+G_4\left(\frac{s}{a},p\right)\sinh(\gamma'_{21}\xi)\sinh(\gamma'_{21}\eta)\right\}\mathrm{d}s, \quad (29)$$

in which the functions $G_i(s,p)$ $(i = 1\text{–}4)$, γ'_{i1} $(i = 1,2)$ are obtained from eqns (A6), (A7) in Appendix B.

We note that the kernel function $K_1(\xi,\eta,p)$ is a semi-infinite integral which has a slow rate of convergence. To evaluate the integral in eqn (28), we consider the contour integrals

$$I_{\Gamma_1} = \frac{-2}{\pi i(\xi\eta)^{1/2}}\oint_{\Gamma_1} M(w,\gamma''_{11},\gamma''_{21})\,\mathrm{e}^{iw\eta}\sin(w\xi)\,\mathrm{d}w \qquad (\xi<\eta),$$
$$I_{\Gamma_2} = \frac{-2}{\pi i(\xi\eta)^{1/2}}\oint_{\Gamma_2} M(w,\gamma''_{11},\gamma''_{21})\,\mathrm{e}^{-iw\eta}\sin(w\xi)\,\mathrm{d}w \qquad (\xi<\eta), \quad (30)$$

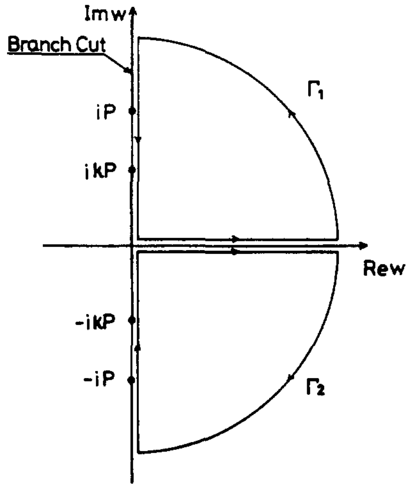

Fig. 2. Contours of integration Γ_1, Γ_2.

where

$$M(w,\gamma''_{11},\gamma''_{21}) = \frac{(w^2+\gamma''^2_{21})^2-4w^2\gamma''_{11}\gamma''_{21}}{2\gamma''_{11}(1-\kappa^2)P^2w} - 1, \tag{31}$$

$$\gamma''_{11} = (w^2+P^2\kappa^2)^{1/2}, \qquad \gamma''_{21} = (w^2+P^2)^{1/2}, \qquad P = \frac{pa}{c_{21}} \tag{32}$$

and the contours Γ_1, Γ_2 are defined in Fig. 2. Since the integrals in eqns (30) satisfy Jordan's lemma on the infinite quarter circles and $I_{\Gamma_1}+I_{\Gamma_2} = 0$, the kernel $K_1(\xi,\eta,p)$ for $\xi < \eta$ can be finally written as

$$K_1(\xi,\eta,P) = \frac{P}{\pi(1-\kappa^2)}\left[\int_0^{\kappa} \frac{(w^2-\hat{\gamma}''^2_{21})^2+4w^2\hat{\gamma}''_{11}\hat{\gamma}''_{21}}{\hat{\gamma}''_{11}w}\right.$$
$$\left.\times e^{-P\eta w}\sinh(P\xi w)\,dw + \int_{\kappa}^{1} 4w\hat{\gamma}''_{21}\,e^{-Pw\eta}\sinh(Pw\xi)\,dw\right] \quad (\xi<\eta), \tag{33}$$

in which the functions $\hat{\gamma}''_{11}$ and $\hat{\gamma}''_{21}$ are

$$\hat{\gamma}''_{11} = (\kappa^2-w^2)^{1/2}, \qquad \hat{\gamma}''_{21} = (1-w^2)^{1/2}. \tag{34}$$

The value of the kernel for $\xi > \eta$ is obtained by interchanging ξ and η in eqn (33).

By making the shear modulus of a surrounding material to infinity, we solve the transient problem of a penny-shaped crack in a cylinder that is bonded to a rigid body. In this case, the $\delta_i(s,p)$ $(i = 1–8)$ in eqns (25) are given by eqns (A9) in Appendix C. We can also obtain the case of a cylinder with a penny-shaped crack by simply taking μ_0 to be zero in eqns (A2)–(A5) in Appendix A.

The dynamic stress intensity factor may be determined by obtaining the asymptotic stress near the crack periphery in the Laplace transform domain and then performing a Laplace inversion. The dynamic singular stresses may be expressed as

$$\begin{aligned}
(\sigma_r)_1(r_1,\vartheta_1,T) &\sim \frac{k_1(T)}{\sqrt{2r_1}}\cos\frac{\vartheta_1}{2}\left(1-\sin\frac{\vartheta_1}{2}\sin\frac{3}{2}\vartheta_1\right),\\
(\sigma_z)_1(r_1,\vartheta_1,T) &\sim \frac{k_1(T)}{\sqrt{2r_1}}\cos\frac{\vartheta_1}{2}\left(1+\sin\frac{\vartheta_1}{2}\sin\frac{3}{2}\vartheta_1\right),\\
(\tau_{rz})_1(r_1,\vartheta_1,T) &\sim \frac{k_1(T)}{\sqrt{2r_1}}\cos\frac{\vartheta_1}{2}\sin\frac{\vartheta_1}{2}\cos\frac{3}{2}\vartheta_1,
\end{aligned} \tag{35}$$

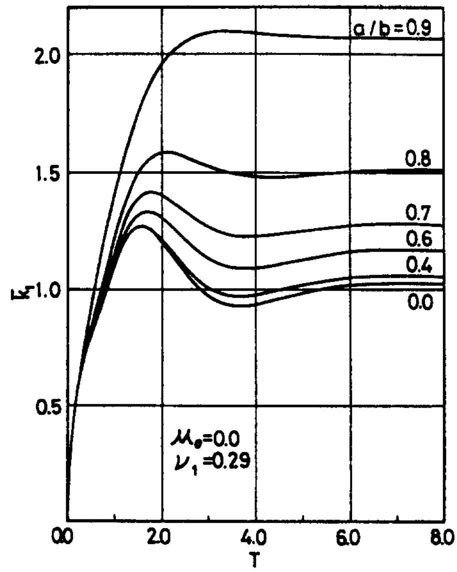

Fig. 3. Dynamic stress intensity factor versus time for $\mu_0 = 0.0$.

where r_1 and ϑ_1 are the polar coordinates defined as

$$r_1 = [(r-a)^2+z^2]^{1/2}, \qquad \vartheta_1 = \tan^{-1}\left[\frac{z}{r-a}\right] \tag{36}$$

and the dynamic stress intensity factor is

$$k_1(T) = \frac{2}{\pi}\sigma_0\sqrt{a}\,\frac{1}{2\pi i}\int_{Br}\frac{\Phi(1,P)}{P}\,e^{PT}\,dP. \tag{37}$$

In eqn (37), $T = c_{21}t/a$ is the nondimensional time. Then a numerical scheme in [5] may be used to evaluate the integral in eqn (37).

NUMERICAL RESULTS AND DISCUSSIONS

Numerical results have been calculated for the dynamic stress intensity factor. The Poisson ratios ν_1, ν_2 and the ratio $\rho_0 = \rho_2/\rho_1$ are taken to be $\nu_1 = \nu_2 = 0.29$ and $\rho_0 = 1.0$. As $T \to \infty$ and $a/b \to 0$, $k_1(T)$ tends to the static solution $(2/\pi)\sigma_0\sqrt{a}$ for a penny-shaped crack in an infinite solid. The stress intensity factors are normalized by $(2/\pi)\sigma_0\sqrt{a}$.

Figure 3 exhibits the variation of the normalized dynamic stress intensity factor $\bar{k}_1 = k_1(T)/(2/\pi)\sigma_0\sqrt{a}$ with the normalized time T for $\mu_0 = 0$ and various a/b ratios. The numerical results for $\mu_0 = 0$ do not coincide with the results by Chen[7]. It appears that this has been caused by the kernel evaluation of the Fredholm integral equation which are not performed by him and the numerical Laplace inversion. As the a/b ratio increases, the peak value of $\bar{k}_1$ also increases and occurs at a later time. Figure 4 shows the results for $\mu_0 = \infty$. It is observed that as the a/b ratio increases, the peak value of $\bar{k}_1$ decreases and occurs at an earlier time. The inertia effect is remarkable for any ratio a/b.

The effect of the μ_0 ratio on $\bar{k}_1$ with T is shown in Figs. 5 and 6 for the ratios $a/b = 0.7$ and 0.8. The peak values of $\bar{k}_1$ are seen to increase and occur at a later time with decreasing values of μ_0. For small μ_0, the inertia effect diminishes and the μ_0 effect becomes dominant.

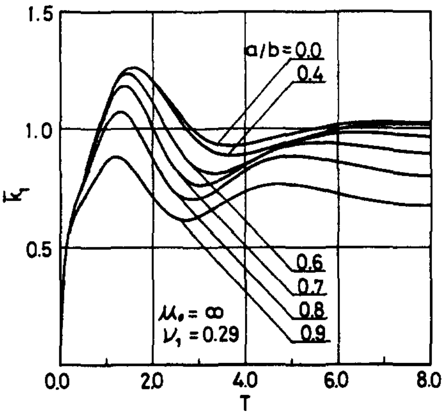

Fig. 4. Dynamic stress intensity factor versus time for $\mu_0 = \infty$.

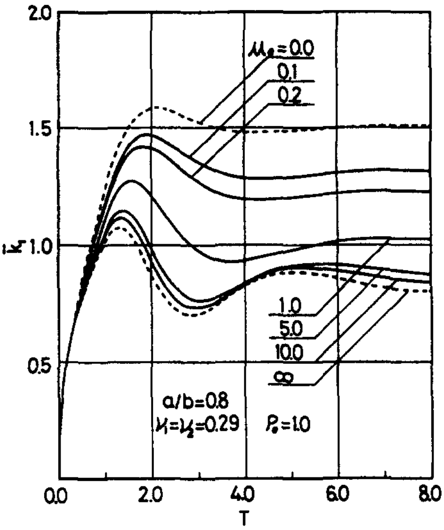

Fig. 5. Dynamic stress intensity factor versus time for $a/b = 0.7$.

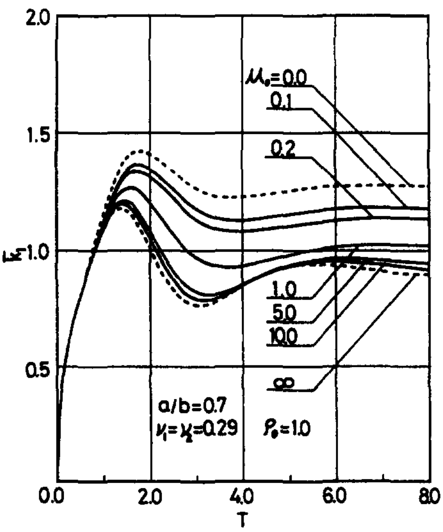

Fig. 6. Dynamic stress intensity factor versus time for $a/b = 0.8$.

The opposite effect is observed for large μ_0. The μ_0 effect for $a/b = 0.8$ is more pronounced than that for $a/b = 0.7$.

The inertia effect for this axisymmetric case is more pronounced than that for the plane case[4].

In summary, the dynamic response of a cylinder composite with a penny-shaped crack under normal impact is determined in this study. The solution is expressed in terms of the dynamic stress intensity factor. The time-dependence of the local stress field is found to depend on the material properties and the geometrical parameters.

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APPENDIX A

The $g_i(\eta,p)$ $(i = 1$–$4)$ and $\delta_i(s,p)$ $(i = 1$–$8)$ in eqns (25) are

$$g_1(\eta,p) = \eta\left(2\eta^2 + \frac{p^2}{c_{21}^2}\right)\left(\frac{1}{\eta^2+\gamma_{11}^2} - \frac{1}{\eta^2+\gamma_{21}^2}\right)J_1(b\eta),$$

$$g_2(\eta,p) = \frac{1}{2}\left[\left(2\eta^2 + \frac{p^2}{c_{21}^2}\right)\frac{\dfrac{p^2}{c_{21}^2} - 2\left(\eta^2 + \dfrac{p^2}{c_{11}^2}\right)}{\eta^2+\gamma_{11}^2} + \frac{4\eta^2\left(\eta^2 + \dfrac{p^2}{c_{21}^2}\right)}{\eta^2+\gamma_{21}^2}\right]J_0(b\eta),$$

$$g_3(\eta,p) = \frac{\eta}{2}\left[\frac{2\eta^2 + \dfrac{p^2}{c_{21}^2}}{\eta^2+\gamma_{11}^2} - \frac{2\left(\eta^2 + \dfrac{p^2}{c_{21}^2}\right)}{\eta^2+\gamma_{21}^2}\right]J_1(b\eta),$$

$$g_4(\eta,p) = \frac{1}{2}\left(\frac{2\eta^2 + \dfrac{p^2}{c_{21}^2}}{\eta^2+\gamma_{11}^2} - \frac{2\eta^2}{\eta^2+\gamma_{21}^2}\right)J_0(b\eta), \tag{A1}$$

$$\delta_1(s,p) = \frac{2}{\pi}\frac{1}{\Delta_2(s,p)}s\alpha_2,$$

$$\delta_2(s,p) = -\frac{2}{\pi}\frac{1}{\Delta_2(s,p)}\alpha_4,$$

$$\delta_3(s,p) = -\frac{2}{\pi}\frac{1}{\Delta_2(s,p)}\frac{2}{b}\alpha_4 + \frac{2}{\pi}\frac{\mu_0}{\Delta_1(s,p)\Delta_2(s,p)}\left[\alpha_4\gamma_{22}K_0(\gamma_{22}b)\right.$$
$$\times\left\{\frac{2}{b}\gamma_{12}K_1(\gamma_{12}b)+(s^2+\gamma_{22}^2)K_0(\gamma_{12}b)\right\}-\alpha_4 s^2 K_0(\gamma_{12}b)\left\{\frac{2}{b}K_1(\gamma_{22}b)\right.$$
$$\left.\left.+2\gamma_{22}K_0(\gamma_{22}b)\right\}-\alpha_2 s\{2\gamma_{12}\gamma_{22}K_1(\gamma_{12}b)K_0(\gamma_{22}b)-(s^2+\gamma_{22}^2)K_1(\gamma_{22}b)K_0(\gamma_{12}b)\}\right],$$

$$\delta_4(s,p) = \frac{2}{\pi}\frac{\mu_0}{\Delta_1(s,p)\Delta_2(s,p)}\left[-\alpha_4 s^2 K_1(\gamma_{22}b)\left\{\frac{2}{b}\gamma_{12}K_1(\gamma_{12}b)\right.\right.$$

$$+(s^2+\gamma_{22}^2)K_0(\gamma_{12}b)\Big\}+\alpha_4 s^2\gamma_{12}K_1(\gamma_{12}b)\left\{\frac{2}{b}K_1(\gamma_{22}b)+2\gamma_{22}K_0(\gamma_{22}b)\right\}$$

$$-\alpha_2 s\gamma_{12}\frac{p^2}{c_{22}^2}K_1(\gamma_{12}b)K_1(\gamma_{22}b)\Big],$$

$$\delta_5(s,p) = -\frac{2}{\pi}\frac{1}{\Delta_2(s,p)}s\alpha_1,$$

$$\delta_6(s,p) = \frac{2}{\pi}\frac{1}{\Delta_2(s,p)}\alpha_3,$$

$$\delta_7(s,p) = \frac{2}{\pi}\frac{1}{\Delta_2(s,p)}\frac{2}{b}\alpha_3 - \frac{2}{\pi}\frac{\mu_0}{\Delta_1(s,p)\Delta_2(s,p)}\Big[\alpha_3\gamma_{22}K_0(\gamma_{22}b)$$

$$\times\left\{\frac{2}{b}\gamma_{12}K_1(\gamma_{12}b)+(s^2+\gamma_{22}^2)K_0(\gamma_{12}b)\right\}-\alpha_3 s^2K_0(\gamma_{12}b)\Big\{\frac{2}{b}K_1(\gamma_{22}b)$$

$$+2\gamma_{22}K_0(\gamma_{22}b)\Big\}-\alpha_1 s\{2\gamma_{12}\gamma_{22}K_1(\gamma_{12}b)K_0(\gamma_{22}b)-(s^2+\gamma_{22}^2)K_1(\gamma_{22}b)K_0(\gamma_{12}b)\}\Big],$$

$$\delta_8(s,p) = -\frac{2}{\pi}\frac{\mu_0}{\Delta_1(s,p)\Delta_2(s,p)}\Big[-\alpha_3 s^2K_1(\gamma_{22}b)\Big\{\frac{2}{b}\gamma_{12}K_1(\gamma_{12}b)$$

$$+(s^2+\gamma_{22}^2)K_0(\gamma_{12}b)\Big\}+\alpha_3 s^2\gamma_{12}K_1(\gamma_{12}b)\left\{\frac{2}{b}K_1(\gamma_{22}b)+2\gamma_{22}K_0(\gamma_{22}b)\right\}$$

$$-\alpha_1 s\gamma_{12}\frac{p^2}{c_{22}^2}K_1(\gamma_{12}b)K_1(\gamma_{22}b)\Big], \tag{A2}$$

where $\alpha_i(s,p)$ $(i = 1\text{–}4)$, $\Delta_i(s,p)$ $(i = 1,2)$ and μ_0 are

$$\alpha_1(s,p) = (s^2+\gamma_{21}^2)I_0(\gamma_{11}b)-\frac{2}{b}\gamma_{11}I_1(\gamma_{11}b)+\frac{\mu_0}{\Delta_1(s,p)}\Big[\Big\{\frac{2}{b}\gamma_{12}K_1(\gamma_{12}b)$$

$$+(s^2+\gamma_{22}^2)K_0(\gamma_{12}b)\Big\}\{s^2K_1(\gamma_{22}b)I_0(\gamma_{11}b)+\gamma_{11}\gamma_{22}K_0(\gamma_{22}b)I_1(\gamma_{11}b)\}$$

$$-\left\{\frac{2}{b}sK_1(\gamma_{22}b)+2s\gamma_{22}K_0(\gamma_{22}b)\right\}\{s\gamma_{11}K_0(\gamma_{12}b)I_1(\gamma_{11}b)$$

$$+s\gamma_{12}K_1(\gamma_{12}b)I_0(\gamma_{11}b)\}\Big],$$

$$\alpha_2(s,p) = -2s\gamma_{21}I_0(\gamma_{21}b)+\frac{2}{b}sI_1(\gamma_{21}b)-\frac{\mu_0}{\Delta_1(s,p)}\Big[\Big\{\frac{2}{b}\gamma_{12}K_1(\gamma_{12}b)$$

$$+(s^2+\gamma_{22}^2)K_0(\gamma_{12}b)\Big\}\{s\gamma_{21}K_1(\gamma_{22}b)I_0(\gamma_{21}b)+s\gamma_{22}K_0(\gamma_{22}b)I_1(\gamma_{21}b)\}$$

$$-\left\{\frac{2}{b}sK_1(\gamma_{22}b)+2s\gamma_{22}K_0(\gamma_{22}b)\right\}\{s^2K_0(\gamma_{12}b)I_0(\gamma_{21}b)$$

$$+\gamma_{12}\gamma_{21}K_1(\gamma_{12}b)I_0(\gamma_{21}b)\}\Big],$$

$$\alpha_3(s,p) = -2s\gamma_{11}I_1(\gamma_{11}b)+\frac{\mu_0}{\Delta_1(s,p)}[2s\gamma_{12}K_1(\gamma_{12}b)\{s^2K_1(\gamma_{22}b)I_0(\gamma_{11}b)$$

$$+\gamma_{11}\gamma_{22}K_0(\gamma_{22}b)I_1(\gamma_{11}b)\}-(s^2+\gamma_{22}^2)K_1(\gamma_{22}b)\{s\gamma_{11}K_0(\gamma_{12}b)I_1(\gamma_{11}b)$$

$$+s\gamma_{12}K_1(\gamma_{12}b)I_0(\gamma_{11}b)\}],$$

$$\alpha_4(s,p) = (s^2+\gamma_{21}^2)I_1(\gamma_{21}b)-\frac{\mu_0}{\Delta_1(s,p)}[2s\gamma_{12}K_1(\gamma_{12}b)\{s\gamma_{21}K_1(\gamma_{22}b)I_0(\gamma_{21}b)$$

$$+s\gamma_{22}K_0(\gamma_{22}b)I_0(\gamma_{21}b)\}-(s^2+\gamma_{22}^2)K_1(\gamma_{22}b)\{s^2K_0(\gamma_{12}b)I_1(\gamma_{21}b)$$

$$+\gamma_{12}\gamma_{21}K_1(\gamma_{12}b)I_0(\gamma_{21}b)\}], \tag{A3}$$

$$\Delta_1(s,p) = \gamma_{12}\gamma_{22}K_1(\gamma_{12}b)K_0(\gamma_{22}b)-s^2K_1(\gamma_{22}b)K_0(\gamma_{12}b),$$
$$\Delta_2(s,p) = \alpha_1\alpha_4-\alpha_2\alpha_3, \tag{A4}$$

$$\mu_0 = \frac{\mu_2}{\mu_1}. \tag{A5}$$

APPENDIX B

The γ'_{11}, γ'_{21} and $G_i(s,p)$ $(i = 1\text{–}4)$ in eqn (29) are

$$\gamma'_{11} = \left(s^2 + \frac{p^2a^2}{c_{11}^2}\right)^{1/2}, \qquad \gamma'_{21} = \left(s^2 + \frac{p^2a^2}{c_{21}^2}\right)^{1/2}, \tag{A6}$$

$$\begin{aligned}
G_1(s,p) = \frac{\alpha}{\gamma_{11}}\left(\frac{p^2}{c_{21}^2} - 2\gamma_{11}^2\right)\Bigg[\delta_1(s,p)\left(\frac{p^2}{c_{21}^2} - 2\gamma_{11}^2\right)K_1(b\gamma_{11})
&+ \delta_2(s,p)\frac{\beta}{\gamma_{11}}K_0(b\gamma_{11}) + \frac{\delta_3(s,p)}{2}\left(\frac{p^2}{c_{21}^2} - 2\gamma_{11}^2\right)K_1(b\gamma_{11}) \\
&+ \frac{\delta_4(s,p)}{2\gamma_{11}}\left(\frac{p^2}{c_{21}^2} - 2\gamma_{11}^2\right)K_0(b\gamma_{11})\Bigg],
\end{aligned}$$

$$\begin{aligned}
G_2(s,p) = 2\alpha s\Bigg[\delta_5(s,p)\left(\frac{p^2}{c_{21}^2} - 2\gamma_{11}^2\right)K_1(b\gamma_{11}) + \frac{\delta_6(s,p)}{\gamma_{11}}\beta K_0(b\gamma_{11})
&+ \frac{\delta_7(s,p)}{2}\left(\frac{p^2}{c_{21}^2} - 2\gamma_{11}^2\right)K_1(b\gamma_{11}) + \frac{\delta_8(s,p)}{2\gamma_{11}}\left(\frac{p^2}{c_{21}^2} - 2\gamma_{11}^2\right)K_0(b\gamma_{11})\Bigg],
\end{aligned}$$

$$\begin{aligned}
G_3(s,p) = \frac{\alpha}{\gamma_{11}}\left(\frac{p^2}{c_{21}^2} - 2\gamma_{11}^2\right)[\delta_1(s,p)(s^2+\gamma_{21}^2)K_1(b\gamma_{21}) + 2\delta_2(s,p)s^2\gamma_{21}K_0(b\gamma_{21})
&+ \delta_3(s,p)s^2K_1(b\gamma_{21}) + \delta_4(s,p)\gamma_{21}K_0(b\gamma_{21})],
\end{aligned}$$

$$\begin{aligned}
G_4(s,p) = 2\alpha s[\delta_5(s,p)(s^2+\gamma_{21}^2)K_1(b\gamma_{21}) + 2\delta_6(s,p)s^2\gamma_{21}K_0(b\gamma_{21})
&+ \delta_7(s,p)s^2K_1(b\gamma_{21}) + \delta_8(s,p)\gamma_{21}K_0(b\gamma_{21})],
\end{aligned} \tag{A7}$$

where

$$\alpha = \frac{c_{21}^2}{p^2(1-\kappa^2)}, \qquad \beta = \frac{p^4}{2c_{21}^4}(1-2\kappa^2) - 2s^2\frac{p^2}{c_{11}^2} - 2s^4. \tag{A8}$$

APPENDIX C

The $\delta_i(s,p)$ $(i = 1\text{–}8)$ in eqns (25) for $\mu_0 = \infty$ are

$$\begin{aligned}
\delta_1(s,p) &= 0, \\
\delta_2(s,p) &= 0, \\
\delta_3(s,p) &= \frac{2}{\pi}\frac{1}{\Delta'_2(s,p)}\gamma_{21}I_0(\gamma_{21}b), \\
\delta_4(s,p) &= \frac{2}{\pi}\frac{1}{\Delta'_2(s,p)}sI_1(\gamma_{21}b), \\
\delta_5(s,p) &= 0, \\
\delta_6(s,p) &= 0, \\
\delta_7(s,p) &= \frac{2}{\pi}\frac{1}{\Delta'_2(s,p)}sI_0(\gamma_{11}b), \\
\delta_8(s,p) &= \frac{2}{\pi}\frac{1}{\Delta'_2(s,p)}\gamma_{11}I_1(\gamma_{11}b),
\end{aligned} \tag{A9}$$

where

$$\Delta'_2(s,p) = \gamma_{11}\gamma_{21}I_1(\gamma_{11}b)I_0(\gamma_{21}b) - s^2I_0(\gamma_{11}b)I_1(\gamma_{21}b). \tag{A10}$$